Gaussian fixed Point and Dangerously Irrelevant Operators

Above we learned that in $d>4$, the fluctuations neglected around the mean-field are negligible. This suggests that one can describe the critical phenomena in $d>4$ by the Gaussian fixed point that neglects the $u\phi^4$ term. This is indeed correct with one major caveat: the naive critical exponents based on the RG eigenvalues y_t, y_h at the Gaussian fixed point yield incorrect critical exponents. This is because the $u\phi^4$ term is <u>dangerously irrelevant</u>. Lets first calculate the naive critical exponents. This will also serve as a check on our calculation above which predicted $\alpha = 2 - d/2$ and $\nu = \frac{1}{2}$ for $d<4$, within the Gaussian approximation.

The Gaussian fixed point, along with the t and h perturbations is:

$$S = \int d^d x \left[\frac{(\nabla\phi)^2}{2} + t\phi^2 + h\phi \right]$$

The simplest way to obtain RG eigenvalues of t, h is to demand that the fixed point action $\int d^d x \frac{(\nabla\phi)^2}{2}$ be dimensionless. Thus ϕ^2 has dimension of $(\text{length})^{-(d-2)}$.

Using this fact, lets introduce dimensionless versions of

t, h by introducing the microscopic length scale 'a' corresponding to the short-distance cutoff ($\sim$ lattice spacing). We will continue to denote the dimensionless versions of t, h by the same symbols to keep the notation simple.

Since the full action is dimensionless, one finds:

t term: $\int d^d x \, a^x \, \phi^2$ dimensionless

$$\Rightarrow d + x - (d-2) = 0 \quad \Rightarrow \quad x = -2$$

h term: $\int d^d x \, a^y \, \phi$ dimensionless

$$\Rightarrow d + y - \frac{(d-2)}{2} = 0 \quad \Rightarrow \quad y = -\left(1 + \frac{d}{2}\right)$$

Thus, in this new notation, the action is:

$$S = \int d^d x \left[\frac{(\nabla\phi)^2}{2} + t\, a^{-2} \phi^2 + h\, a^{-(1+\frac{d}{2})} \phi \right]$$

To obtain the RG flow, one simply rescales $a \to ba$ and demand that t and h transform so that the action is invariant.

$$\Rightarrow \quad t \to t' = b^2 t$$
$$h \to h' = b^{(1+d/2)} h$$

$$\Rightarrow y_t = 2, \quad y_h = (1 + d/2).$$

Therefore, the RG flow of t and h at the Gaussian fixed point is:

$$\frac{dt}{dl} = 2t$$

$$\frac{dh}{dl} = \left(1 + \frac{d}{2}\right) h$$

Another way to derive the RG flow / RG eigenvalues is to recall the relation between scaling dimension x and RG eigenvalue y: $y = d - x$.

At the Gaussian fixed point,

$$\langle \varphi(x)\, \varphi(0) \rangle \sim \int_k \langle |\varphi(k)|^2 \rangle e^{ik.x}$$

$$\sim \int \frac{d^dk}{k^2} e^{ik.x} \sim \frac{1}{x^{d-2}}$$

$\Rightarrow$ scaling dimension $x_\varphi = (d-2)/2$

$$\Rightarrow \quad y_h = d - \frac{(d-2)}{2} = 1 + d/2$$

Similarly $\langle \varphi^2(x)\, \varphi^2(0) \rangle \sim \dfrac{1}{x^{2(d-2)}}$ (homework: verify this)

$$\Rightarrow \quad y_t = d - (d-2)$$
$$= 2.$$

One may now use these RG eigenvalues to obtain the (naive) critical exponents:

Gaussian Fixed Point		Comparison with mean-field / Landau theory
$\alpha = 2 - d/y_t = 2 - \frac{d}{2}$	✗	0
$\beta = (d - y_h)/y_t = \frac{d-2}{4}$	✗	$\frac{1}{2}$
$\gamma = \frac{2y_h - d}{y_t} = 1$	✓	1
$\delta = \frac{y_h}{d - y_h} = \frac{d+2}{d-2}$	✗	3
$\nu = 1/y_t = \frac{1}{2}$	✓	$\frac{1}{2}$
$\eta = d - 2y_h + 2 = 0$	✓	0

We find clear discrepencies between the two sets.
The reason is that to obtain spontaneous sym. breaking, the 'u' term is essential and several quantities depend in a singular way on u.
For example, the magnetization $m \sim \sqrt{-\frac{t}{u}}$ within

the Landau theory, and therefore setting $u=0$ at the outset (as in the Gaussian action) is rather dangerous. Similarly, free energy scales as $-\frac{r^2}{2u}$ within Landau theory and is again singular as $u \to 0$.

To treat such 'dangerously irrelevant operators' (i.e. operators whose RG eigenvalue is negative at the RG fixed point of interest, but on which various quantities depend in a singular way close to the fixed point), let's consider a general setting where

(i) There exist an irrelevant variable 'v' with RG eigenvalue $y_v < 0$.

(ii) A physical quantity of interest, A, is a singular function of v as $A \sim \frac{1}{v^{\sigma}}$.

Lets also allow for one relevant variable t at this critical point with RG eigenvalue $y_t > 0$.

For example, the free energy in $d > 4$ scales as $f_s \sim -\frac{1}{u}$ so that $\sigma = 1$. Therefore, one can write

$$f_s(t, u) = b^{-d} f_s(b^{y_t} t, b^{y_u} u)$$

where $y_t = 2$ and $y_u = 4 - d < 0$.

Returning to the more general discussion, we have

$$A(t, v) = b^{y_A} A(b^{y_t} t, b^{y_v} v)$$

Setting $t \, b^{y_t} \sim 1$

$$A(t, v) \sim t^{-y_A/y_t} \phi(v \, t^{-y_v/y_t})$$

where ϕ is as yet unknown scaling f^n.

Now, we demand that $A \sim \frac{1}{v^\sigma}$

$$\Rightarrow \phi(x) \sim \frac{1}{x^\sigma}$$

$$\Rightarrow A(t, v) \sim \frac{t^{-y_A/y_t}}{v^\sigma \, t^{-\sigma y_v/y_t}}$$

$$\sim \frac{t^{1/y_t(-y_A - \sigma |y_v|)}}{v^\sigma}$$

(recall $|y_v| = -y_v$ since $y_v < 0$)

$$\Rightarrow \quad \boxed{A(t, v) \sim t^{-(y_A + \sigma |y_v|)/y_t}}$$

Lets consider the case of our immediate interest – the scaling free energy for $d > 4$. Recall that naive (incorrect) scaling argument gave exponent $\alpha = 2 - \frac{d}{2}$ $\Rightarrow$ $f_s \sim t^{2-\alpha} \sim t^{d/2}$.

To take into account the dangerous irrelevance of u, as just discussed, the exponents are

$$y_t = 2, \quad \sigma = 1, \quad y_u = 4 - d, \quad y_{f_s} = -d.$$

$$\Rightarrow \quad f_s(t, u) \sim t^{-[-d + (d-4) \times 1]/2}$$

$$\sim t^2$$

$\Rightarrow$ true specific heat exponent $\alpha = 2 - 2 = 0$, which matches with the mean-field theory (and disagrees with the naive scaling $\alpha = 2 - \frac{d}{2}$).

Let's consider another example, namely, that of magnetization m. Within mean-field theory,

$m \sim \sqrt{-\frac{t}{u}}$, so that the exponent $\beta = \frac{1}{2}$.

However, naive scaling at the Gaussian fixed point yields $\beta = \frac{d-2}{4}$. This is again due to the singular dependence of m on u.

Let's start by writing,

$$f_s(t, u, h) = b^{-d} f_s(b^{y_t} t, b^{y_h} h, b^{y_u} u)$$

$$m = \frac{\partial f_s}{\partial h}\Big|_{h=0} = b^{-d+y_h} f_s(b^{y_t} t, 0, b^{y_u} u)$$

$$\Rightarrow y_m = -d + y_h = -d + \frac{d}{2} + 1 = 1 - \frac{d}{2}$$

Further, since $m \sim \frac{1}{\sqrt{u}} \Rightarrow \sigma = \frac{1}{2}$. Using

$y_t = 2$ and $y_u = 4 - d \Rightarrow$

$$m \sim |t|^{-[y_m + \sigma |y_u|]/y_t}$$

$$\sim |t|^{-[1 - \frac{d}{2} + \frac{1}{2}(d-4)]/2}$$

$$\sim |t|^{1/2} \Rightarrow \text{true } \beta = 1/2 .$$